Asymptotic Variance of Sample Vector Variance of Standardized Variables

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Abstract

The stability of the correlation matrices is noteworthy. Usually to testing stability of coorelation matrices used to statistics M Box, Jennrich and G. Its statistics However, M Box and G statistics as computation of matrix determinant and J statistic involves matrix inversion. The former needs the condition that all sample correlation matrices are positive definite which is not always satisfied in practice. This condition is not apt for high dimension data sets because its computational efficiency becomes low. To handle this obstacles, we proposed a new statistical test based on what we call vector variance of standardized variables (VVSV). The proposed test is constucted based on vector variance (VV).This is evidenced by several papers describing the correlation matrix, Vector variance of standardized variables sample used a statistical formula variance vector. In practice there are difficulties in the calculation to determine variance of Vector Variance of Sample Variance of Standardized Variables. In this paper, by utilizing the vec operator and properties of the matrix will be investigated alternative formulation of asymptotic variance of Vector Variance of Sample Variance of Standardized Variables.

Keywords: vec operator, vector variance, covariance matrix, correlation matrix

1. Introduction

The stability of correlation matrix is one of the most important problems in economic development and financial industry. We can see in the literature that, since the last decade, that problem can be found in a wide spectrum of applications. Its applications in property, real estate and asset businesses can be seen, for example, in eichholtz (1995), Lee(1998), Cooka el al. (2002) and Fischer (2007). Other applications such as in equity market, global market and risk management are presented by Meric and meric (1997), Tang (1998), Gande and Parsley (2002), Annaert et al. (2003), Ragea (2003), Da costa et al. (2005) and Goetzmann et al. (2005), Michelle et al. (2010). We can also find its application in parallel computation of high dimensional robust correlation matrices in Chilson et al. (2006).

Djauhari and Herdiani (2010) proposed a new statistical test based on what we call vector variance of standardized variables (VVSV). The proposed test is constucted based on vector variance (VV). The role of VV could be increasing the computational efficiency of fast minimum covariance determinant (FMCD) algorithm, Herwindiati et al. (2008). Let P be the correlation matrix of the population under studi. Vector Variance of Standardized Variable (VVSV) is the trace of the squared correlation matrix population, i.e. $VVSV = Tr\left(P^2\right) = \left\|vec(P)\right\|^2$.

In practice there are difficulties in calculating variance of vector variance of standardized variables. Therefore, in this paper we investigate about on asymptotic variance of vector variance of standardized variables. The rest of the paper is organized as follows. In section 2, we discuss on the Asymptotic Variance of Sample Vector Variance of Standardized Variables. Our approach will be based on the notions of vec operator and commutation matrix the problem formulation will be presented. Later on, in section 3, Additional remarks will close this paper.

2. Asymptotic Variance of Sample Vector Variance of Standardized Variables

Let X is a random vector p dimension with definite positive covariance matrixΣ. By using vec operator, see Muirhead (1982), El Maache and Lepage (1998), Schott (1997, 2007) and Djauhari (2007), vector variance (VV) of X is simply $\|vec(\Sigma)\|^2$. It is a multivariate variability measure like Generalized Variance (GV). See Djauhari (2007) for an in depth discussion on VV and its asymtotic behavior. In what follows our attention will be focused on the case where all variables are standardized.

Let Z be the random vector where its k-th component is the standardized version of the k-th component of X; k=1, 2, ...,p. The covariance matrix P of Z is the correlation matrix of X. We call the parameter $\|vec(\mathrm{P})\|^2$ vector variance of the standardized variables (VVSV). Now, let $Z_1, Z_2, \dots, Z_n$ be a random sample of size n from Z with covariance matrix P. If R is the sample correlation matrix, then we call $\|vec(\mathrm{R})\|^2$ sample VVSV.

Let $\vec{Z}_1, \vec{Z}_2, \cdots, \vec{Z}_n$ be a random sample of size n from the distribution having covariance matrix P. If R be a sample correlation matrix then $vec(R)$ is a representation of R in vector form. Neudecker (1996) and Schott (2007) wrote about asymptotic distribution of $vec(R)$ as follows.

Theorem 1

Let $\vec{X}_1, \vec{X}_2, \cdots, \vec{X}_n$ be a random vector of size n from the distribution normal p-variat, $\vec{\boldsymbol{X}} \sim N_p(\vec{\mu}, \boldsymbol{\Sigma})$ then

$$\sqrt{n-1}\{vec(R) - vec(P)\} \xrightarrow{d} N_{p^2}(\vec{0}, \Gamma)$$

Where :

a. $\Gamma = 2M_p \Phi M_p$, $M_p = \frac{1}{2}\left(I_{p^2} + K_{pp}\right)$
b. K_{pp} is commutation matrix of size $p^2 \times p^2$
c. $\Phi = \{I_{p^2} - (I_p \otimes P)\Lambda_p\}(P \otimes P)\{I_{p^2} - \Lambda_p(I_p \otimes P)\}$
d. $\Lambda_p = \sum_{i=1}^{p}\left(\vec{e}_i\vec{e}_i^t \otimes \vec{e}_i\vec{e}_i^t\right)$, $\vec{e}_i$ is i-th column of identity matrix I_p of size $p \times p$.

Value approximation of $vec\ (R)$, as follows

$$vec(R) \approx vec(P + A - \tfrac{1}{2}\ (PD_A + D_A P))$$

It's approximation will be used to find mean and variance of $vec\ (R)$. Mean of *vec(R) is* vec(P) and variance of $\mathrm{var}(vec(R))$ $\left\{M_p\left[I_{p^2} - \frac{1}{2}\{(I_p \otimes P) + (P \otimes I_p)\}\Lambda_p\right]^t\right\}$

Cause $\left[I_{p^2} - \frac{1}{2}\{(I_p \otimes P) + (P \otimes I_p)\}\Lambda_p\right] M_p = M_p \{I_{p^2} - (I_p \otimes P)\Lambda_p\}$, and M_p symmetry, see Schott (1997, p. 415 and 282), then

$\mathrm{var}(vec(R)) \approx$

$$\approx \frac{2}{n-1} M_p \left\{ I_{p^2} - (I_p \otimes P)\Lambda_p \right\} (P \otimes P) \left\{ I_{p^2} - \Lambda_p (I_p \otimes P) \right\} M_p$$

Therefore,

$$\left\{ I_{p^2} - (I_p \otimes P)\Lambda_p \right\} (P \otimes P) \left\{ I_{p^2} - \Lambda_p (I_p \otimes P) \right\} = \Phi$$

We can get,

$$var(vec(R)) \approx \frac{2}{n-1} M_p \, \Phi \, M_p$$

Further, Let $\vec{Z}_1, \vec{Z}_2, \cdots, \vec{Z}_n$ be a random sample of size n from the distribution having covariance matrix P. If R be a sample correlation matrix then $vec(R)$ is a representation of R in vector form with dimension p^2. Thus, VVSV sample be a $\|vec(\boldsymbol{R})\|^2$ or $Tr(\boldsymbol{R}^2)$, It is number of elements diagonal matrix of $\boldsymbol{R}^2$, namely

$$Tr(\boldsymbol{R}^2) = \sum_{i=1}^{p} \sum_{j=1}^{p} r_{ij}^2$$

Asymptotic distribution of $\|vec(\boldsymbol{R})\|^2$ can be written as follows.

Theorem 2

If $\sqrt{n-1}\{vec(\boldsymbol{R}) - vec(\boldsymbol{P})\} \xrightarrow{d} N_{p^2}(\vec{\boldsymbol{0}}, \boldsymbol{\Gamma})$ and $(vec(\boldsymbol{R}))^t(vec(\boldsymbol{R})) = \|vec(\boldsymbol{R})\|^2$ then $\|vec(R)\|^2 \xrightarrow{d} N(\eta, \theta^2)$, with mean $\eta = \|vec(\mathrm{P})\|^2$ and variance $\theta^2 = \frac{8}{n-1}(vec(\mathrm{P}))^t M_p \Phi M_p (vec(\mathrm{P}))$

Where :

$P =$ Population correlation matrix

$R =$ Sample correlation matrix

$p =$ Number of variable

$M_p = \frac{1}{2}\left(I_{p^2} + K_{pp}\right)$, K_{pp} be a commutation matrix with size $p^2 \times p^2$

$\Phi = \left\{I_{p^2} - \left(I_p \otimes P\right)\Lambda_p\right\}(P \otimes P)\left\{I_{p^2} - \Lambda_p\left(I_p \otimes P\right)\right\}$

(Djauhari & Erna 2008)

Formulation variance of VVSV was been caused difficultly to compute, therefore its need an alternative formulation by vec operator and commutation matrix to become simply. The result is written in theorem form, as follows.

Proposition

If $var\{(\|vec(\mathrm{R})\|^2)\} = \frac{8}{n-1}(vec(\mathrm{P}))^t M_p \Phi M_p (vec(\mathrm{P}))$, then $(vec(\mathrm{P}))^t M_p \Phi M_p (vec(\mathrm{P})) = Tr(P^4) - 2Tr\left(D_{p^2} P^3\right) + Tr(D_{p^2} P^3)^2$

Proof:

$(vec(\mathrm{P}))^t M_p \Phi M_p (vec(\mathrm{P})) =$

$$= (vec(P))^t \frac{1}{2}\left(I_{p^2} + K_{pp}\right) \left\{ I_{p^2} - (I_p \otimes P)\Lambda_p \right\} (P \otimes P) \left\{ I_{p^2} - \Lambda_p (I_p \otimes P) \right\} \frac{1}{2}\left(I_{p^2} + K_{pp}\right)(vec(P))$$

$$= \frac{1}{4}(vec(P))^t \left(I_{p^2} + K_{pp}\right) \left\{ I_{p^2} - (I_p \otimes P)\Lambda_p \right\} (P \otimes P) \left\{ I_{p^2} - \Lambda_p (I_p \otimes P) \right\} \left(I_{p^2} + K_{pp}\right)(vec(P))$$

$$= \frac{1}{4}\left((vec(P))^t I_{p^2} + (vec(P))^t K_{pp}\right) \left\{ I_{p^2} - (I_p \otimes P)\Lambda_p \right\} (P \otimes P) \left\{ I_{p^2} - \Lambda_p (I_p \otimes P) \right\}$$

$\left(I_{p^2}(vec(P)) + K_{pp}(vec(P))\right)$ where K_{pp} be a symmetry

$$= \frac{1}{4}\left((vec(P))^t + (vec(P))^t\right)\left\{I_{p^2} - \left(I_p \otimes P\right)\Lambda_p\right\}(P \otimes P)\left\{I_{p^2} - \Lambda_p\left(I_p \otimes P\right)\right\}$$

$$\left((vec(P)) + vec\left(P^t\right)\right), \; K_{pp}\, vec(P) = vec\,(P^t)$$

$$= \frac{1}{4}\left((vec(P))^t + (vec(P))^t\right)\left\{I_{p^2} - \left(I_p \otimes P\right)\Lambda_p\right\}(P \otimes P)\left\{I_{p^2} - \Lambda_p\left(I_p \otimes P\right)\right\}$$

$((vec(P)) + vec(P))$, P be a matrix symmetry

$$= \frac{1}{4}\left(2\,(vec(P))^t\right)\left\{I_{p^2} - \left(I_p \otimes P\right)\Lambda_p\right\}(P \otimes P)\left\{I_{p^2} - \Lambda_p\left(I_p \otimes P\right)\right\}(2vec(P))$$

$$= \left\{(vec(P))^t I_{p^2} - (vec(P))^t \left(I_p \otimes P\right)\Lambda_p\right\}(P \otimes P)\left\{I_{p^2}\, vec(P) - \Lambda_p\left(I_p \otimes P\right)vec(P)\right\}$$

$$= \left\{(vec(P))^t (P \otimes P) - (vec(P))^t \left(I_p \otimes P\right)\Lambda_p (P \otimes P)\right\}\left\{vec(P) - \Lambda_p\left(I_p \otimes P\right)vec(P)\right\}$$

$$= Tr\left(P^4\right) - (vec(P))^t \left(I_p^t \otimes P^t\right)\Lambda_p (P \otimes P) vec(P) -$$

$$= Tr\left(P^4\right) - (vec(P))^t \left(I_p \otimes P\right)^t \Lambda_p (P \otimes P) vec(P) -$$

$$(vec(P))^t (P \otimes P)^t \Lambda_p \left(I_p \otimes P\right) vec(P) + (vec(P))^t \left(I_p \otimes P\right)\Lambda_p (P \otimes P)\Lambda_p \left(I_p \otimes P\right) vec(P)$$

$$= Tr\left(P^4\right) - \left(vec\left(PPI_p\right)\right)^t \Lambda_p\, vec\left(P^3\right) -$$

$$\left(vec\left(P^3\right)\right)^t \Lambda_p\, vec\left(PPI_p\right) + \left(vec\left(PPI_p\right)\right)^t \Lambda_p\,(P \otimes P)\,\Lambda_p\, vec\left(PPI_p\right)$$

$$= Tr\left(P^4\right) - \left(vec\left(P^2\right)\right)^t \Lambda_p\, vec\left(P^3\right) -$$

$$\left(vec\left(P^3\right)\right)^t \Lambda_p\, vec\left(P^2\right) + \left(vec\left(P^2\right)\right)^t \Lambda_p\,(P \otimes P)\,\Lambda_p\, vec\left(P^2\right)$$

$$= Tr\left(P^4\right) - \left(vec\left(P^2\right)\right)^t \Lambda_p^t\, vec\left(P^3\right) -$$

$$\left(vec\left(P^3\right)\right)^t \Lambda_p\, vec\left(P^2\right) + \left(vec\left(P^2\right)\right)^t \Lambda_p^t\,(P \otimes P)\,\Lambda_p\, vec\left(P^2\right),\; sebab\, \Lambda_p^t = \Lambda_p$$

$$= Tr\left(P^4\right) - \left(vec\left(D_{p^2}\right)\right)^t vec\left(P^3\right) - \left(vec\left(P^3\right)\right)^t vec\left(D_{p^2}\right) + \left(vec\left(D_{p^2}\right)\right)^t (P \otimes P)\, vec\left(D_{p^2}\right)$$

$$= Tr\left(P^4\right) - Tr\left(D_{p^2}^t\, P^3\right)^t - Tr\left(\left(P^3\, D_{p^2}\right)\right) + \left(vec\left(D_{p^2}\right)\right)^t (P \otimes P)\, vec\left(D_{p^2}\right)$$

$$= Tr\left(P^4\right) - Tr\left(D_{p^2}\, P^3\right) - Tr\left(P^3\, D_{p^2}\right) + \left(vec\left(D_{p^2}\right)\right)^t (P \otimes P)\, vec\left(D_{p^2}\right)$$

Selanjutnya,

karena $\left(vec\left(D_{p^2}\right)\right)^t (P \otimes P)\, vec\left(D_{p^2}\right) = Tr\left(D_{p^2} P D_{p^2} P\right)$ lihat persamaan (2.6) sehingga,

$$= Tr\left(P^4\right) - Tr\left(D_{p^2}\, P^3\right) - Tr\left(P^3\, D_{p^2}\right) + Tr\left(D_{p^2} P D_{p^2} P\right)$$

karena $D_{p^2} P D_{p^2} P = \left(D_{p^2} P\right)^2$

$$= Tr\left(P^4\right) - Tr\left(D_{p^2}\, P^3\right) - Tr\left(P^3\, D_{p^2}\right) + Tr\left(D_{p^2} P\right)^2$$

P^3 dan D_{p^2} simetri maka $D_{p^2}P^3$ juga simetri sehingga

$$= Tr\left(P^4\right) - 2Tr\left(D_{p^2} P^3\right) + Tr\left(D_{p^2} P\right)^2$$

Akhirnya terbukti bahwa

$$(vec\,(P))^t M_p\, \Phi\, M_p\, (vec(P)) = Tr\left(P^4\right) - 2Tr\left(D_{p^2} P^3\right) + Tr\left(D_{p^2} P\right)^2$$

3. Additional Remarks

Alternative formulation of variance of VVVS is

$$\frac{8}{n-1}\left\{Tr(P^4) - 2Tr\left(D_{p^2}P^3\right) + Tr(D_{p^2}P)^2\right\}$$

References

Annaert, J., Claes, A.G.P., dan De Custer, A.J.K. (2003). Inter Temporal Stability of the European Credit Spread Co-Mvement Structure. University of Antwerp, Faculty of Applied Economics in its series Working papers.

Chilson, J., Ng, R., Wagner, A. dan Zamar, R. (2006). Parallel Computation of High Dimensional Robust Correlation and Covariance Matrices. *Algorithmica, Springer* New York, 45(3) 403-431.

Cooka, W., Mounfield, C., Ormerod, P., dan Smith, L. (2002). Eigen analysis of the stability and degree of information content in correlation matrices contructed from property time series data. *The European Physical Journal*, B, 27, 189-195.

Da Costa, Jr, N., Nunes, S., Caretta, P., dan Da Silva, S. (2005). Stock-Market co-movements revisited. *Economics Bulletin*, 7(3), 1-9.

Djauhari, M.A. (2007). A. Measure of Data Concentration. *Journal of Probability and Statistics*, vol 2, No. 2, 139-135.

Djauhari, M.A. dan Herdiani, E.T. (2008). Monitoring the Stability of Correlation Stucture in Drive Rib Production Process: An MSPC Approach. *The Open Industrial and Manufacturing Eigneering Journal*, Vol. 1, 8-18.

Djauhari, M.A. dan Herdiani, E.T. (2010). *A.Test For Stability of Correlation Matrix. International Conference on Quantitative Methods Used in Economics and Business,* Universitas Malahayati. 83-92.

Eichholtz, P.M.A. (1995). The Stability of Covariance of International Proferty Share Returns. *The Journal of Real Estat Research*, Accepted.

El Maache, H. dan Lepage, Y. (1998). Mesures d'Association Vectorielle Basées sur une Matrice de Corrélation. *Revue de Statistique Appliquée*, 46(4), 27-43.

Fischer, M. (2007) *Are Correlations Constant Over Time? Application of the CC-TRIG- Test to Retum Series from Different Asset Classes*, SFB 649 Disscussion Paper 2007-012, University of Erlangen-Numberg, Germany.

Gande, A. dan Parsley, D.C. (2002). *News Spillovers in the Sovereign Debt Market*. Forthcoming Journal of Financial Economics

Goetzmann, W.N., Li., L., dan Rouwenhorst, K.G. (2005). *Long-Term Global Market Correlation*. Journal of Business, 78(1), 1-38.

Herdiani E.T. (2008). Statistik Penguji Kestabilan Barisan Matriks Korelasi, Disertasi, Institut Teknologi Bandung.

Herwindiati, D.E., Djauhari, M.A., dan Mashuri, M (2007). Robust Mulitivariate Outlier Labeling, *Communication in Statistics*: *Computation and simulation.* 36(6), 1287-1294.

Jennrich, R.I. (1970). An Asymptotic Test for the Equality of Two Correlation Matrices. *Journal of the American Statistical Association*, 65(330), 904-912.

Lee, S.L. (1998). The Inter-Temporal Stability of Real Estate Returns: An Empirical Investigation. *International Real Estate Conference*. Maastricht, the Netherlands, June 10-13.

Meric, I. dan Meric, G. (1997). Co-Movement of European Equity Markets Before and After the 1987 Crash*, *Multinational Finance Journal*, vol. 1, no. 2, 137-152

Michele, T., Fabrizio, L., Rosario, N., M., Correlation, hierarchies, and networks in financial markets, Journal of Economic Behavior & Organization 75 (2010) 40–58

Muirhead, R.J. (1982). *Aspect of Multivariate Statistical Theory*. John Wiley & Sons, Inc., New York.

Neudecker, H. (1996). The asymptotic variance matrices of the sample correlation matrix in elliptical and normal situations and their proportionality, *Linear Algebra and Its Applications*, 237/238:127-132

Ragea, V. (2003). Testing Correlation Stability During Hectic Financial Markets. *Financial Market and Portfolio Management*, 17(3), 289-308.

Schott, J.R. (1997). *Matrix Analysis for Statistics*. John Wiley & Sons, New York.

Schott, J.R. (2007). Testing the Equality of Correlation Matrices when Sample Correlation Matrices are Dependent. *Journal of Statistical Planning and Inference*, 137, 1992-1997.
Sindelar, M.F., (2007). Multivariate statistical process control, Disertation, University of Pittsburgh
Tang, G.Y.N. (1998). The Intertemporal Stability of the Covariance and Correlation Matrices of Hong Kong Stock Returns. *Applied Financial Economics*, 8, 359-365.